

# Brane cosmology in teleparallel and $f(T)$ gravity

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## Abstract

We consider the cosmology of a brane-world scenario in the framework of teleparallel and  $f(T)$  gravity in a way that matter is localized on the brane. We show that the cosmology of such branes is different from the standard cosmology in teleparallelism. In particular, we obtain a class of new solutions with a constant five-dimensional radius and cosmologically evolving brane in the context of constant torsion  $f(T)$  gravity.

## 1 Introduction

Einstein had the idea of unifying gravitation and electromagnetism in 1928 [1]. This attempt was based on the mathematical structure of teleparallelism, also referred to as distant or absolute parallelism. In other words, the idea was the introduction of a tetrad field — a field of orthonormal bases on the tangent spaces at each point of the four-dimensional space-time. The tetrad has 16 components whereas the gravitational field, represented by the space-time metric, has only 10. The six additional degrees of freedom of the tetrad was then supposed by Einstein to be related to the six components of the electromagnetic field [1]. This attempt of unification did not succeed, because the additional six degrees of freedom of the tetrad are actually eliminated by the six-parameter local Lorentz invariance of the theory. However, Einstein introduced concepts that remain important to the present day. Teleparallelism can be considered by using the Weitzenböck connection, which has torsion, rather than the curvature defined by the Levi-Civita connection [2]. The teleparallel Lagrangian density is described by the torsion scalar, i.e,  $T$ . Recently, the authors have extended the Lagrangian density of teleparallel gravity, the so-called  $f(T)$  gravity, in which various gravitational and cosmological solutions of this model are studied in [3]-[9]. This concept is similar to the idea of  $f(R)$  gravity. Some people have studied brane world scenario in the framework of extended theories of gravitation such as  $f(R)$  gravity [10]. Thus, inspired by these theories we became interested in studying brane cosmology within the teleparallelism theory.

Moreover, the idea that our world might be a brane embedded in a higher-dimensional space-time (the bulk) [11] has been in the mainstream of cosmological investigations in the past few years [12]. This approach differs from the usual Kaluza-Klein idea in that the size of the extra dimensions can be large. The concept of large extra dimensions is discussed phenomenologically in [13]. An important ingredient of the brane world scenario is that matter is confined to the

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brane and the only communication between the brane and the bulk is through gravitational interaction or some other dilatonic matter. In general, the matter on the brane leads to a cosmological evolution which is different from the usual evolution governed by the Friedmann equation, that is, in brane cosmology the Hubble parameter on the brane is proportional to the square of energy density [14]. This proportionality is a result of the application of the Israel matching condition which is basically a relation between the extrinsic curvature and the energy-momentum tensor representing matter fields on the brane. Extra dimension models of teleparallel and  $f(T)$  gravity have recently been studied in [15]. In the present work, we study the teleparallel and  $f(T)$  gravity in five-dimensions.

The organization of the paper is as follows: in section 2 we briefly review the teleparallel and  $f(T)$  gravity in five-dimensions and write the full system of equations. In section 3 we consider the cosmological equations for teleparallel and  $f(T)$  gravity by imposing a constant torsion condition on the solutions. In the section 3.1 we study brane equations by inserting tension on the brane. Finally, we study the solutions with a constant five-dimensional radius in section 4. Conclusions are drawn in the last section.

## 2 Brane cosmology in teleparallel and $f(T)$ gravity

We consider a curvature-free brane embedded in a five-dimensional space-time (the bulk). We assume that our brane is located at  $y = 0$ . In teleparallel gravity the tetrad components  $e_A(x^\mu)$  are the fundamental structures of the theory, where an index  $A$  runs over  $0, 1, 2, 3, 4$  for the tangent space at each point  $x^\mu$  of the manifold. The relationship between tetrad and the space-time metric is given by

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \quad (1)$$

where  $\mu$  and  $\nu$  are Lorentzian (coordinates) indices on the manifold and run over  $0, \dots, 4$ , and  $\eta_{AB} = \text{diag}[-1, 1, 1, 1, 1]$ .

In teleparallel gravity one uses the curvature-less Weitzenböck connection  $\overset{w}{\Gamma}{}^\rho_{\nu\mu} \equiv e_A{}^\rho \partial_\mu e^A_\nu$  [2] thus according to this connection the torsion  $T^\rho_{\mu\nu}$  and contorsion  $K^{\mu\nu}_\rho$  tensors are, respectively, given by

$$T^\rho_{\mu\nu} = \overset{w}{\Gamma}{}^\rho_{\nu\mu} - \overset{w}{\Gamma}{}^\rho_{\mu\nu} = e_A{}^\rho (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu), \quad (2)$$

$$K^{\mu\nu}_\rho = -\frac{1}{2}(T^{\mu\nu}_\rho - T^{\nu\mu}_\rho - T^\rho_{\mu\nu}). \quad (3)$$

By using the above equations one can define the torsion scalar  $T$  as follows

$$T = S_\rho{}^{\mu\nu} T^\rho_{\mu\nu}, \quad (4)$$

in which

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}_\rho + \delta_\rho{}^\mu T^{\alpha\nu}_\alpha - \delta_\rho{}^\nu T^{\alpha\mu}_\alpha). \quad (5)$$

In the context of 5D  $f(T)$  gravity, we can write the Lagrangian in terms of torsion scalar, as [4]

$$I = \frac{1}{2\kappa_5^2} \int d^5x \, e \, f(T), \quad (6)$$

where  $e = |e| = \det(e^A_\mu) = \sqrt{-\det g_{\mu\nu}}$  and  $\kappa_5^2 = 8\pi G_5$ .<sup>1</sup> In teleparallel gravity all gravitational fields are considered in the torsion tensor  $T^\rho_{\mu\nu}$ , and torsion scalar,  $T$ , comes from it in a similar way as the curvature scalar,  $R$ , arises from the curvature (Riemann) tensor.

Variation of the action (6), with respect to tetrad, gives the equations of motion [4]

$$f_T [\partial_\mu (e e_A^\rho S_\rho^{\nu\mu}) - e e_A^\lambda S^{\rho\mu\nu} T_{\rho\mu\lambda}] + f_{TT} e e_A^\lambda S_\lambda^{\nu\mu} \partial_\mu T + \frac{1}{2} e e_A^\nu f(T) = \Theta_A^\nu, \quad (7)$$

where  $f_T = \partial f(T)/\partial T$ ,  $f_{TT} = \partial^2 f(T)/\partial T^2$  and  $\Theta_A^\nu$  is the energy-momentum tensor of the perfect fluid.

On the other hand, from the relation between the Weitzenböck connection and the Levi-Civita connection given by equation (3), one can write the Riemann tensor for the Levi-Civita connection in the form

$$\begin{aligned} R^\rho_{\mu\lambda\nu} &= \partial_\lambda \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\mu\lambda} \\ &= \nabla_\nu K^\rho_{\mu\lambda} - \nabla_\lambda K^\rho_{\mu\nu} + K^\rho_{\sigma\nu} K^\sigma_{\mu\lambda} - K^\rho_{\sigma\lambda} K^\sigma_{\mu\nu}, \end{aligned} \quad (8)$$

whose associated Ricci tensor can then be written as

$$R_{\mu\nu} = \nabla_\nu K^\rho_{\mu\rho} - \nabla_\rho K^\rho_{\mu\nu} + K^\rho_{\sigma\nu} K^\sigma_{\mu\rho} - K^\rho_{\sigma\rho} K^\sigma_{\mu\nu}. \quad (9)$$

Now, using  $K^\rho_{\mu\nu}$  given by equation (3) along with the relations  $K^{(\mu\nu)\sigma} = T^{\mu(\nu\sigma)} = S^{\mu(\nu\sigma)} = 0$  and considering that  $S^\mu_{\rho\mu} = 2K^\mu_{\rho\mu} = -2T^\mu_{\rho\mu}$  one can get [16]-[19]

$$\begin{aligned} R_{\mu\nu} &= -\nabla^\rho S_{\nu\rho\mu} - g_{\mu\nu} \nabla^\rho T^\sigma_{\rho\sigma} - S^{\rho\sigma}_\mu K_{\sigma\rho\nu}, \\ R &= -T - 2\nabla^\mu T^\nu_{\mu\nu}, \end{aligned} \quad (10)$$

and thus can obtain

$$G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = -\nabla^\rho S_{\nu\rho\mu} - S^{\sigma\rho}_\mu K_{\rho\sigma\nu}, \quad (11)$$

where  $G_{\mu\nu} = R_{\mu\nu} - (1/2) g_{\mu\nu} R$  is the Einstein tensor.

Finally, by using equation (11), the field equations for  $f(T)$  gravity equation (7) can be rewritten in the form [16]

$$G_{\mu\nu} + \frac{1}{2f_T} (f(T) - T f_T) g_{\mu\nu} + B_{\mu\nu} \frac{f_{TT}(T)}{f_T} = \frac{1}{f_T} \Theta_{\mu\nu}, \quad (12)$$

where we have defined  $B_{\mu\nu} = S_{\nu\mu}^\sigma \nabla_\sigma T$ . When  $f(T) = T$ , general relativity is recovered, which verifies the claim that teleparallel gravity and general relativity are equivalent. In this case the field equations are clearly covariant and the theory is also local Lorentz invariant. In the more general case with  $f(T) \neq T$  however, this is not the case.

For considering the cosmology of the model, we take the five-dimensional metric as follows<sup>2</sup>

$$ds^2 = -A^2(t, y) dt^2 + B^2(t, y) d\mathbf{x}^2 + C^2(t, y) dy^2, \quad (13)$$

where  $d\mathbf{x}^2 = dx_1^2 + dx_2^2 + dx_3^2$ . With regard to the above relation,  $g_{\mu\nu} = \text{diag}[-A^2(t, y), B^2(t, y), B^2(t, y), B^2(t, y), C^2(t, y)]$  and then using equation (1), the diagonal tetrad components read as

$$e^A_\mu = \text{diag}[A(t, y), B(t, y), B(t, y), B(t, y), C(t, y)]. \quad (14)$$

<sup>1</sup>We have set units  $8\pi G_5 = 1$ .

<sup>2</sup>Note that the spatial curvature of the three-dimensional metric,  $k$ , is considered to be zero (spatially flat).

Now, by substituting equation (14) into the relation (2) and with the help of equations (3)-(5), one can obtain the value of the torsion scalar  $T$  as follows<sup>3</sup>:

$$T = \frac{6}{A^2 B^2 C^2} \left[ -(AA')(BB') + C^2 \dot{B}^2 - A^2 B'^2 + (B\dot{B})(C\dot{C}) \right], \quad (15)$$

where a dot denotes derivative with respect to  $t$ . In order to realize the  $Z_2$  symmetry, the coefficients  $A(t, y)$ ,  $B(t, y)$  and  $C(t, y)$  depend on  $y$  through its modulus  $|y|$ . To survey the cosmological setup, we take the  $\Theta_\mu^\nu$  as follows

$$\Theta_\mu^\nu = \frac{1}{C(y, t)} \text{diag}[-\rho_b, p_b, p_b, p_b, 0] \delta(y), \quad (16)$$

where  $\rho_b := \rho_b(t)$  is the brane energy density and  $p_b := p_b(t)$  the brane pressure. With this choice, the matter is localized on the brane. Furthermore, we require the equation of state of the matter on the brane to have the following form

$$p_b = \omega \rho_b, \quad (17)$$

where  $\omega$  is a real constant.

To continue, we study  $f(T)$  gravity in the five-dimensions in the context of constant torsion regime. In this way, the equation of motion (12) reduces to

$$\mathcal{E}_{\mu\nu} \equiv G_{\mu\nu} + \frac{1}{2f_T} (f(T) - T f_T) g_{\mu\nu} = \frac{1}{f_T} \Theta_{\mu\nu}. \quad (18)$$

Thus, the non-vanishing components of the equation of motion are  $\mathcal{E}_{00}, \mathcal{E}_{11} = \mathcal{E}_{22} = \mathcal{E}_{33}, \mathcal{E}_{44}$  and  $\mathcal{E}_{04}$  that are, respectively, given by

$$\begin{aligned} \frac{1}{C f_T} \rho_b \delta(y) &= 3 \left[ -\frac{1}{BC^2} (B'' + 2B' \delta(y)) + \frac{B'}{BC^2} \left( \frac{C'}{C} - \frac{B'}{B} \right) + \frac{\dot{B}}{BA^2} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \right] \\ &\quad + \frac{1}{2} \left( T - \frac{f(T)}{f_T} \right), \end{aligned} \quad (19)$$

$$\begin{aligned} \frac{1}{C f_T} p_b \delta(y) &= \left[ \frac{2}{BC^2} (B'' + 2B' \delta(y)) + \frac{1}{AC^2} (A'' + 2A' \delta(y)) + \frac{B'}{BC^2} \left( \frac{B'}{B} - \frac{2C'}{C} \right) \right. \\ &\quad + \frac{A'}{AC^2} \left( \frac{2B'}{B} - \frac{C'}{C} \right) - \frac{\dot{B}}{BA^2} \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) + \frac{\dot{A}}{A^3} \left( \frac{\dot{C}}{C} + \frac{2\dot{B}}{B} \right) \\ &\quad \left. - \frac{\ddot{C}}{CA^2} - \frac{2\ddot{B}}{BA^2} \right] - \frac{1}{2} \left( T - \frac{f(T)}{f_T} \right), \end{aligned} \quad (20)$$

$$0 = 3 \left[ \frac{B'}{B} \left( \frac{B'}{B} + \frac{A'}{A} \right) + \frac{C^2 \dot{B}}{BA^2} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) - \frac{C^2 \ddot{B}}{BA^2} \right] - \frac{1}{2} C^2 \left( T - \frac{f(T)}{f_T} \right), \quad (21)$$

$$0 = 3 \left( \frac{\dot{B}A'}{BA} + \frac{B'\dot{C}}{BC} - \frac{\dot{B}'}{B} \right) (2\theta(y) - 1). \quad (22)$$

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<sup>3</sup>With regard to the mentioned notation in [20],  $\frac{\partial A}{\partial y} = A' \frac{d|y|}{dy} = A'(2\theta(y) - 1)$ ,  $(\frac{\partial A}{\partial y})(\frac{\partial B}{\partial y}) = A'B'$  and  $\frac{\partial^2 A}{\partial y^2} = A'' + 2A'\delta(y)$ , where  $\theta(y)$  is the Heaviside function;  $A'$  and  $A''$  denote, respectively, the first and the second derivative of  $A$  with respect to  $|y|$ . Note that  $A''$  is the non-distributional part of the double derivative of  $A$  (the standard derivative) in which it vanishes on the brane.

With regard to matching the delta function on both sides of the first two equations ((19) and (20)), we then obtain

$$\frac{B'_0}{B_0} = -\frac{1}{6f_T}C_0\rho_b, \quad \frac{A'_0}{A_0} = \frac{1}{6f_T}C_0(2\rho_b + 3p_b), \quad (23)$$

where  $A_0 := A(t, 0)$ ,  $B_0 := B(t, 0)$  and  $C_0 := C(t, 0)$ . Once this matching is carried out, the delta function contributions cancel out and the equations become valid everywhere. Also notice that the obtained equation of state is not of the form  $p_b = \omega\rho_b$  but a time-dependent one.

### 3 Teleparallel and $f(T)$ brane equations

To obtain the Friedmann-like equation [13] in constant torsion  $f(T)$  gravity, we first introduce the function [20]

$$F(t, y) = \frac{(BB')^2}{C^2} - \frac{(B\dot{B})^2}{A^2}, \quad (24)$$

then, by assuming that equation (22) is satisfied, the components  $\mathcal{E}_{00}$  and  $\mathcal{E}_{44}$  of the equation of motion can be written in the following form

$$F'(t, y) = \frac{B^3 B'}{3} \left( T - \frac{f(T)}{f_T} \right), \quad \dot{F}(t, y) = \frac{B^3 \dot{B}}{3} \left( T - \frac{f(T)}{f_T} \right). \quad (25)$$

One can integrate the above equations and deduce the first integral of motion as

$$\frac{(BB')^2}{C^2} - \frac{(B\dot{B})^2}{A^2} = \frac{1}{12}B^4 \left( T - \frac{f(T)}{f_T} \right) + \mathcal{C}, \quad (26)$$

where  $\mathcal{C}$  is an integration constant. Thus, using equation (26), the function  $A$  is entirely determined in such a way that it is, in terms of  $B$ ,  $C$  and their derivatives, given by [20]

$$A^2 = \dot{B}^2 \left[ \frac{B'^2}{C^2} - \frac{1}{12}B^2 \left( T - \frac{f(T)}{f_T} \right) - \frac{\mathcal{C}}{B^2} \right]^{-1}. \quad (27)$$

Finally, by evaluating the above equation at  $y = 0$  together with the use of the matching conditions (23), imposing the equation of state (17), and considering the temporary gauge  $A_0 = 1$ , the Friedmann-like equation [13] in constant torsion  $f(T)$  gravity is obtained to be of the form

$$H^2 = \frac{\dot{B}_0^2}{B_0^2} = \frac{1}{36f_T^2}\rho_b^2 - \frac{\Lambda}{12} - \frac{\mathcal{C}}{B_0^4}, \quad (28)$$

where  $\Lambda = \left( T - \frac{f(T)}{f_T} \right)$ . Next, we must examine the conservation of matter on the brane in teleparallel gravity. Thus, by imposing the matching conditions on the  $\mathcal{E}_{04}$  at  $y = 0$  and taking  $p_b = \omega\rho_b$ , one gets the conservation equation as

$$\dot{\rho}_b + 3(1 + w)\rho_b H = 0. \quad (29)$$

Having solved the above equation, we obtain the energy density as follows

$$\rho_b = \rho_0 B_0^{-3(1+w)}, \quad (30)$$

where  $\rho_0$  is an integration constant. Using the Friedmann-like equation (28) with  $\mathcal{C} = 0$  and equation (29) we find

$$\dot{\rho}_b^2 = \frac{(1+\omega)^2}{4} \rho_b^2 \left( \frac{\rho_b^2}{f_T^2} - 3\Lambda \right). \quad (31)$$

Here, we study  $f(T)$  gravity in the context of constant torsion regime. Therefore, we must impose a constant torsion condition in our study. In this manner we take  $T = T_0 = \text{const.}$  Now, using equations (28) and (29) and then inserting the matching conditions (23) into the right-hand side of equation (15) as a constant value  $T_0$ , one can get on the brane

$$\frac{\dot{C}_0}{C_0} = \frac{(1+\omega)\rho_b}{2\dot{\rho}_b} \left( \frac{(2+3\omega)}{6f_T} \rho_b^2 - \frac{\Lambda}{2} - T_0 \right). \quad (32)$$

To elaborate on our study, here, we will consider the cases corresponding to  $\Lambda = 0$ ,  $\Lambda < 0$  and  $\Lambda > 0$  on the brane. In this way, the scale factor on the brane and the deceleration parameter are calculated for all cases.

*Case i)* For the choice  $\Lambda = 0$ , i.e.,  $f(T) = T$ , the energy density on the brane as a function of the cosmic time is

$$\rho_b = - \left( \beta_0 \pm \frac{(1+w)}{2} t \right)^{-1}, \quad (33)$$

where  $\beta_0$  is an integration constant. By putting the above equation into the equation (30) we find the scale factor on the brane as follows:

$$B_0(t) \sim \left( \beta_0 \pm \frac{(1+w)}{2} t \right)^{\frac{1}{3(1+w)}}, \quad (34)$$

From the above equation we deduce

$$B_0(t) \sim t^{\frac{1}{3(1+w)}}, \quad (35)$$

thus, the accelerated brane universe occurred once the equation of state satisfied  $w < -\frac{2}{3}$ ; this is the equation of state for dark energy.

To probe our model in the cosmological background, we look at the behavior of the deceleration parameter on the brane. The deceleration parameter,  $q$ , is enumerated as

$$q = - \frac{B_0 \ddot{B}_0}{\dot{B}_0^2}, \quad (36)$$

where  $B_0$  is scale factor on the brane. By using equation (34), the equation (36) can be written as

$$q = 3w + 2. \quad (37)$$

In the accelerating universe,  $q$  is negative. To perceive the deceleration parameter, we plot  $q$  as a function of  $w$  in Figure 1. From the above discussion, it can be seen that the obtained results for  $f(T) = T$  correspond to teleparallel equivalence to general relativity, and the field equations reduce to the Einstein equations. In addition, in this case ( $\Lambda = 0$ ) the solution of equation (32) is found to be of the form

$$C_0(t) = \nu_0 \left( \beta_0 \pm \frac{1+\omega}{2} t \right)^{-\frac{(2+3\omega)}{3(1+\omega)}} e^{\pm T_0 (\beta_0 \pm \frac{1+\omega}{4} t) t}, \quad (38)$$

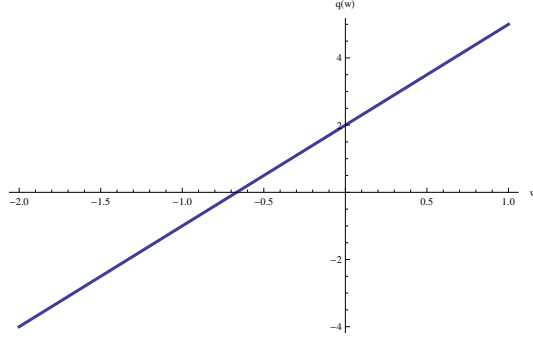


Figure 1: The behavior of deceleration parameter  $q$  as a function of  $w$ . An accelerating universe occurs for  $w < -\frac{2}{3}$ .

where  $\nu_0$  is an integration constant.

*Case ii)* For  $\Lambda < 0$  we introduce  $\Lambda = -\eta^2$ . In this case, the energy density on the brane from equation (31) as a function of the cosmic time is given by

$$\rho_b = \pm \frac{\sqrt{3}\eta f_T}{\sinh\left(\frac{\sqrt{3}(1+\omega)\eta t}{2}\right)}. \quad (39)$$

For the above solution, the scale factor on the brane takes the following form

$$B_0(t) \sim \left( \frac{e^{\frac{\sqrt{3}(1+\omega)\eta t}{2}} - e^{-\frac{\sqrt{3}(1+\omega)\eta t}{2}}}{2} \right)^{\frac{1}{3(1+\omega)}}. \quad (40)$$

Inserting equation (40) into equation (36), the deceleration parameter can be cast in the form

$$q = \frac{12(1+\omega) - \left(2 + e^{\sqrt{3}(1+\omega)\eta t} + e^{-\sqrt{3}(1+\omega)\eta t}\right)}{\left(2 + e^{\sqrt{3}(1+\omega)\eta t} + e^{-\sqrt{3}(1+\omega)\eta t}\right)}. \quad (41)$$

It can be seen that for  $t \ll 1$ , equation (37) is recovered and this situation corresponds to  $\Lambda = 0$ . For  $t \gg 1$  we can get  $q = -1$ , thus, in the late time we have the accelerating universe (eternal de Sitter universe). Also, to calculate  $C_0(t)$  one must substitute equation (39) into (32) together with  $\Lambda = -\eta^2$ .

*Case iii)* For the case corresponding to  $\Lambda > 0$  we put  $\Lambda = \eta^2$ . Similar to the above, we obtain

$$B_0(t) \sim \sec^{-\frac{1}{3(1+\omega)}} \left( \gamma_0 \pm \frac{\sqrt{3}(1+\omega)\eta}{2} t \right), \quad (42)$$

where  $\gamma_0$  is a constant of integration. For this case, the deceleration parameter is given by

$$q + 1 = 3(1+\omega) \csc^2 \left( \gamma_0 \pm \frac{\sqrt{3}(1+\omega)\eta}{2} t \right). \quad (43)$$

Again, one can use equations (30), (32) and (42) to obtain  $C_0(t)$  for this case.

Before proceeding to study the brane equations with tension, we obtain a condition for accelerated expansion on the brane. We show that in the limit  $\rho_b^2 \gg \frac{3}{2}\Lambda f_T^2$ , there is an accelerating universe. To this end, we first write the component  $\mathcal{E}_{44} = 0$  of equation of motion at the position of the brane,  $y = 0$ . Then, by using the matching conditions (23) and the normalization  $A_0 = 1$ , we arrive at

$$\frac{\ddot{B}_0}{\dot{B}_0} + \frac{\dot{B}_0^2}{B_0^2} = -\frac{1}{36f_T^2}\rho_b(\rho_b + 3p_b) - \frac{\Lambda}{6}. \quad (44)$$

Subtracting equation (28) with  $\mathcal{C} = 0$  from equation (44) gives

$$\frac{\ddot{B}_0}{\dot{B}_0} = -\frac{1}{6f_T^2}\left[\frac{1}{3}\rho_b^2 + \frac{1}{2}\rho_b p_b + \frac{1}{2}\Lambda f_T^2\right]. \quad (45)$$

Thus the condition for accelerated expansion on the brane is

$$\ddot{B}_0 > 0, \quad \text{if} \quad p_b < -\left(\frac{2\rho_b^2 + 3\Lambda f_T^2}{3\rho_b}\right). \quad (46)$$

In the limit  $\rho_b^2 \gg \frac{3}{2}\Lambda f_T^2$  we have an accelerating universe if  $p_b < -\frac{2}{3}\rho_b$ .

### 3.1 The brane equations with tension

In this subsection we shall consider a brane with total energy density  $\rho_b = \rho_m + \lambda$ , with  $\rho_m$  being the energy density of the matter on the brane and  $\lambda$  the constant tension of the brane. By considering the cosmic matter as a perfect fluid with equation of state  $p_m = \omega\rho_m$  where  $p_m = p_b + \lambda$ , equation (29) is expressed

$$3\dot{B}_0(\rho_m + p_m) + B_0\dot{\rho}_m = 0, \quad (47)$$

for which we have a solution similar to (30). In the presence of  $\rho_m$  and  $\lambda$ , equation (28) with  $\mathcal{C} = 0$  takes the form

$$H^2 = \frac{1}{18f_T^2}\left(\frac{1}{2}\rho_m^2 + \rho_m\lambda\right) + \bar{\Lambda}, \quad (48)$$

where  $\bar{\Lambda} = \frac{\lambda^2}{36f_T^2} - \frac{\Lambda}{12}$ . By introducing a new variable  $x = B_0^{\bar{q}}$  in which  $\bar{q} = 3(1 + \omega)$ , equation (48) is written as

$$\dot{x}^2 = \bar{q}^2\left(\bar{\Lambda}x^2 + \frac{\lambda\bar{\rho}_0}{18f_T^2}x + \frac{\bar{\rho}_0^2}{36f_T^2}\right). \quad (49)$$

where  $\bar{\rho}_0$  is an integration constant in equation (47). In order to complete the study of the cases corresponding to  $\bar{\Lambda} = 0$ ,  $\bar{\Lambda} > 0$  and  $\bar{\Lambda} < 0$ , we explore the kinds of cosmology associated with the scale factor in (49). Case  $\bar{\Lambda} = 0$  means that there are some constant  $f(T)$ 's which are satisfied in the following equation

$$Tf_T^2 - f(T)f_T = \frac{\lambda^2}{3}.$$

In this case ( $\bar{\Lambda} = 0$ ) with the initial condition  $B_0(0) = 0$  for equation (49) we have the following solution

$$B_0^{\bar{q}}(t) = \frac{\bar{q}\bar{\rho}_0}{6f_T}\left(\frac{\bar{q}\lambda}{12f_T}t^2 + t\right). \quad (50)$$



For cases  $\bar{\Lambda} > 0$  and  $\bar{\Lambda} < 0$ , integration of equation (49) with  $B_0(0) = 0$  gives

$$B_0^{\bar{q}}(t) = \frac{\bar{\rho}_0}{6f_T\sqrt{\bar{\Lambda}}} \sinh\left(\bar{q}\sqrt{\bar{\Lambda}}t\right) + \frac{\bar{\rho}_0\lambda}{36f_T^2\bar{\Lambda}} \left[\cosh\left(\bar{q}\sqrt{\bar{\Lambda}}t\right) - 1\right], \quad \bar{\Lambda} > 0, \quad (51)$$

$$B_0^{\bar{q}}(t) = \frac{\bar{\rho}_0}{6f_T\sqrt{|\bar{\Lambda}|}} \sin\left(\bar{q}\sqrt{|\bar{\Lambda}|}t\right) - \frac{\bar{\rho}_0\lambda}{36f_T^2|\bar{\Lambda}|} \left[\cos\left(\bar{q}\sqrt{|\bar{\Lambda}|}t\right) - 1\right], \quad \bar{\Lambda} < 0. \quad (52)$$

If we choose  $f(T) = T^n$ , the constant torsion will be depended on  $\lambda$ . The results have been summarized in Table 1.

Table 1: Constraints on  $T$  and  $n$  for cases  $\bar{\Lambda} > 0$  and  $\bar{\Lambda} < 0$  when  $f(T) = T^n$ .

$f(T) = T^n$	$T^{2n-1} < \frac{\lambda^2}{3(n^2-n)}$	$T^{2n-1} > \frac{\lambda^2}{3(n^2-n)}$
$\bar{\Lambda} > 0$	$n \in (-\infty, 0) \cup (1, \infty)$	$0 < n < 1$
$\bar{\Lambda} < 0$	$0 < n < 1$	$n \in (-\infty, 0) \cup (1, \infty)$

## 4 Solutions with a constant five-dimensional radius

We consider the five-dimensional solution of the model by assuming the scale factor of the five-dimensional to be constant and normalized to 1 at all times. We find a class of solutions with vanishing bulk matter and without a cosmological constant on the bulk. In this respect, with  $C(t, y) = 1$ ,  $\mathcal{E}_{04} = 0$  leads to

$$\frac{A'}{A} = \frac{\dot{B}'}{\dot{B}}. \quad (53)$$

Integration gives

$$\dot{B} = Ag(t), \quad (54)$$

where  $g(t)$  is an arbitrary function of  $t$ . Note that  $g(t) = \dot{B}_0$  since  $A_0 = 1$ . Furthermore, by inserting relations (53) and (54) into the component  $\mathcal{E}_{00}$  of the equation of motion, we find the following equation

$$(B^2)'' - \frac{1}{3}\Lambda B^2 = 2g^2(t). \quad (55)$$

As explained in section 3, we study  $f(T)$  gravity in the context of constant torsion regime. Thus, to obtain the solutions of the model with  $C(t, y) = 1$  we must also impose constant torsion condition on the solutions. By imposing that the torsion scalar is a constant and by considering equation (15) with  $C(t, y) = 1$ , we then get

$$T_0 = 6 \left( -\frac{A'B'}{AB} + \frac{\dot{B}^2}{A^2B^2} - \frac{B'^2}{B^2} \right). \quad (56)$$

Then, by using the condition (23) and regarding  $H^2 = \frac{\dot{B}_0^2}{B_0^2}$ , with  $H^2$  given by equation (28) (with  $\mathcal{C} = 0$ ), we obtain the following equation

$$\frac{1}{6f_T^2}(2 + 3\omega)\rho_b^2 - \frac{\Lambda}{2} - T_0 = 0. \quad (57)$$

From the above equation  $\rho_b$  is found to be of the form

$$\rho_b^2 = \frac{(3T_0 f_T - f(T))f_T}{(\omega + \frac{2}{3})}. \quad (58)$$

We note that in this case  $\rho_b$  is constant and thus this equation restricts the solutions. As an example, for  $\omega = -\frac{2}{3}$  one can obtain  $f(T) = T^{\frac{1}{3}}$ . Now, by solving equation (55) we obtain the solutions of the model for the cases corresponding to  $\Lambda = 0$ ,  $\Lambda > 0$  and  $\Lambda < 0$ .

• As mentioned in the preceding section, the case where  $\Lambda = 0$  corresponds to  $f(T) = T$ . For this case, integration with respect to  $y$  and the use of equation (54) gives

$$B^2(t, y) = \zeta(t)|y| + g^2(t)y^2 + B_0^2, \quad (59)$$

where  $\zeta(t)$  is an arbitrary function of  $t$ . One can use the first equation of (23) with  $C_0 = 1$  to determine the function  $\zeta(t)$ . Thus, utilizing equation (54) together with equation (29) leads to

$$B^2(t, y) = B_0^2 \left(1 - \frac{\rho_b}{3}|y|\right) + g^2(t)y^2, \quad (60)$$

$$A(t, y) = \frac{B_0}{B} \left[1 + \frac{(\rho_b + 3p_b)}{3}|y|\right] + \frac{\dot{g}(t)}{B}y^2. \quad (61)$$

By substituting solutions (60) and (61) into equation (56) one can get  $g^2(t)$  on the brane as follows:

$$g^2(t) = \frac{B_0^2}{6} \left(T_0 - \frac{(1 + 3\omega)}{3}\rho_b^2\right), \quad (62)$$

where  $\rho_b$  is given by the equation (58).

• For the case  $\Lambda > 0$  we take  $\Lambda = \eta^2$ . Then, equation (55) is written as

$$(B^2)'' - \frac{1}{3}\eta^2 B^2 = 2g^2(t). \quad (63)$$

Thus, by solving equation (63) one can get

$$B^2(y, t) = \psi(t)e^{\frac{\eta}{\sqrt{3}}|y|} + \phi(t)e^{-\frac{\eta}{\sqrt{3}}|y|} - \frac{6}{\eta^2}g^2(t), \quad (64)$$

where  $\psi(t)$  and  $\phi(t)$  are arbitrary functions of  $t$ . One can use the first equation of (23) with  $C_0 = 1$  to determine the functions  $\psi(t)$  and  $\phi(t)$ . Finally, by using equations (54) and (29) we obtain [21]

$$B^2(y, t) = \left(B_0^2 + \frac{6}{\eta^2}g^2(t)\right) \cosh\left(\frac{\eta|y|}{\sqrt{3}}\right) - \frac{B_0^2\rho_b}{\sqrt{3}\eta f_T} \sinh\left(\frac{\eta|y|}{\sqrt{3}}\right) - \frac{6}{\eta^2}g^2(t), \quad (65)$$

$$A(y, t) = \frac{1}{B(y, t)} \left[ \left(B_0 + 6\dot{g}(t)/\eta^2\right) \cosh(\eta|y|/\sqrt{3}) + B_0 \left(\frac{3p_b + \rho_b}{2\sqrt{3}\eta f_T}\right) \sinh(\eta|y|/\sqrt{3}) - \frac{6}{\eta^2}\dot{g}(t) \right], \quad (66)$$

where  $\rho_b$  is given by the equation (58). Also, to calculate  $g(t)$  one must use equations (65), (66) and (56) on the brane. For the case  $\Lambda = \eta^2 > 0$  when we choose  $f(T) = T^n$ , then,  $T$  is positive if  $n \in (-\infty, 0) \cup (1, \infty)$  and  $T$  is negative if  $0 < n < 1$ .

• By taking  $\Lambda = -\eta^2$  for case  $\Lambda < 0$ , the general solution of equation (55) is found to be of the form

$$B^2(y, t) = \bar{\psi}(t) \cos\left(\frac{\eta|y|}{\sqrt{3}}\right) + \bar{\phi}(t) \sin\left(\frac{\eta|y|}{\sqrt{3}}\right) + \frac{6}{\eta^2}g^2(t), \quad (67)$$

where  $\bar{\psi}(t)$  and  $\bar{\phi}(t)$  are arbitrary functions of  $t$ . Similar to the previous cases, we get

$$B^2(y, t) = \left( B_0^2 - \frac{6}{\eta^2} g^2(t) \right) \cos\left(\frac{\eta|y|}{\sqrt{3}}\right) - \frac{B_0^2 \rho_b}{\sqrt{3} \eta f_T} \sin\left(\frac{\eta|y|}{\sqrt{3}}\right) + \frac{6}{\eta^2} g^2(t), \quad (68)$$

$$A(y, t) = \frac{1}{B(y, t)} \left[ \left( B_0 - 6\dot{g}(t)/\eta^2 \right) \cos(\eta|y|/\sqrt{3}) + B_0 \left( \frac{3p_b + \rho_b}{2\sqrt{3} \eta f_T} \right) \sin(\eta|y|/\sqrt{3}) + \frac{6}{\eta^2} \dot{g}(t) \right]. \quad (69)$$

Similarly, one can substitute the above equations into (56) to obtain  $g(t)$  for this case.

## 5 Conclusion

In this paper, we have discussed  $f(T)$  gravity in five dimensions within the context of a constant torsion regime. We have considered the cosmological equations for teleparallel and  $f(T)$  gravity on the brane. We are aiming to further understand the behavior of the brane universe studied for the cases corresponding to  $\Lambda = 0$ ,  $\Lambda < 0$  and  $\Lambda > 0$  on the brane. In *case i*) we have shown that  $f(T) = T$  corresponds to teleparallel equivalence to general relativity and the field equations reduce to the Einstein equations. For *case i*) ( $\Lambda = 0$ ) the deceleration parameter as a function of  $w$  has been plotted. From Figure 1, it can be seen that an accelerating universe occurs for  $w < -\frac{2}{3}$ . We have reached the result that, in the limit  $t \ll 1$ , the case  $\Lambda < 0$  corresponds to  $\Lambda = 0$  and, for  $t \gg 1$ , the deceleration parameter approaches -1, namely  $q \rightarrow -1$ . Furthermore, we have found a condition for accelerated expansion on the brane in the limit of  $\rho_b^2 \gg \frac{3}{2} \Lambda f_T^2$ , so an accelerating universe occurs provided that  $p_b < -\frac{2}{3} \rho_b$ . Moreover, the scale factor on the brane in the presence of tension is obtained. By imposing that the torsion scalar be a constant and by considering equation (15) with  $C(t, y) = 1$ , we have shown that for  $\omega = -\frac{2}{3}$ , one possibility is  $f(T) = T^{\frac{1}{3}}$ . Finally, by imposing constant torsion condition on the solutions we have obtained a class of solutions in the bulk in which the fifth dimension does not evolve dynamically.

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